

ON SOME CLASSES OF BANACH SPACES AND
GENERALIZED HARMONIC ANALYSIS

by

Ka-Sing Lau*

§1. Introduction.

In classical harmonic analysis, two types of functions f are considered: (i) f can be represented as a trigonometric series, i.e.

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k t}, \quad t \in \mathbb{R},$$

(ii) $f \in L^2(\mathbb{R})$. Then by the Inversion theorem,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(\omega) e^{i\omega t} d\omega.$$

In terms of optics, the spectrum of the light signal f in (i) is made up of finite or countable number of sharp lines of intensity $|c_k|^2$ at the frequency ω_k . The light signal f in (ii) has continuous spectrum on the frequency band \mathbb{R} . Note that in this case, for each fixed $h > 0$,

$$\lim_{t \rightarrow \pm\infty} \int_t^{t+h} |f(\tau)|^2 d\tau = 0.$$

*This research was partially supported by NSF-MCS-7903638.

This means that the energy emitted by the signal during a time interval of fixed length h approaches 0 as the interval advances to infinity on the time axis.

In the early twenty century, some physicists such as Rayleigh, Schuster, Taylor, were interested on the type of white light signals f (e.g. sunlight) that have continuous spectrum and infinite energy (i.e. $\int_{-\infty}^{\infty} |f(t)|^2 dt = \infty$). The two classical approaches do not seem to explain the behavior of such light signals satisfactorily. Wiener felt that the difficulty stemmed from the limitation of the classical theory. Around the twenties, he began developing a "generalized" harmonic analysis that could cover signals f on \mathbb{R} which are on one hand so irregular that their spectrum are not made up of sharp lines alone and on the other so lastingly vigorous that

$\int_t^{t+h} |f(\tau)|^2 d\tau \neq 0$ as $t \rightarrow \infty$. The class of functions $W^2(\mathbb{R})$ he considers is the set of Borel measurable functions f on \mathbb{R} such that

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(t)|^2 dt$$

exists and is finite [19]. In order to study the spectrum of the functions $f \in W^2(\mathbb{R})$ and the covariance function

$$\phi(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t+\tau) \overline{f(t)} dt,$$

Wiener introduced the following integrated Fourier transformation $W(f)$ of f defined by

$$g(u) = \frac{1}{2\pi} \left(\int_{-\infty}^{-1} + \int_1^{\infty} \frac{f(t)e^{-itu}}{-it} dt + \int_{-1}^1 f(t) \frac{e^{-itu}-1}{-it} dt \right). \quad (1.1)$$

(The last term on the right hand side of (1.1) guarantees integrability about the origin). Analogous to the Plancherel theorem in the L^2 case, he showed that for $f \in W^2(\mathbb{R})$

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(t)|^2 dt = \lim_{h \rightarrow 0^+} \frac{1}{2h} \int_{-\infty}^{\infty} |g(u+h) - g(u-h)|^2 du. \quad (1.2)$$

It has been found that the theory of generalized harmonic analysis is applicable to diverse areas of pure and applied mathematics. In particular, it was used to consider the problem of anti-aircraft fire control with radar during World War II and brought into the theory of prediction and filtering. For a detail account of this and its relationship with Kolmogorov's stochastic process, the reader may refer to [1], [4], [10], [12], [15].

The class of functions $W^2(\mathbb{R})$ defined above is, however, not closed under addition, hence many functional analytic techniques are not applicable in the theory. To remedy this, Masani developed a nonlinear Banach graph theory to study $W^2(\mathbb{R})$ and its closed subspaces [14], [15]. Yet another approach is to embed $W^2(\mathbb{R})$ into a larger Banach space; a suitable one will be the Marcinkiewicz space $M^2(\mathbb{R})$ [11]. This space had been considered by Bohr and Følner [3] and Bertrandias [2]. It is the purpose of the paper to report some recent results of the joint work of Lee and the author on this direction. The detail will appear elsewhere ([6], [7], [8]).

§2. The Marcinkiewicz Spaces.

For $1 \leq p < \infty$, let $M^p(\mathbb{R})$ denote the class of complex valued Borel measurable functions f on \mathbb{R} such that

$$\|f\|_{M^p} = \overline{\lim}_{T \rightarrow \infty} \left(\frac{1}{2T} \int_{-T}^T |f|^p \right)^{\frac{1}{p}} < \infty$$

and let $W^p(\mathbb{R})$ be the set of $f \in M^p(\mathbb{R})$ such that

$$\lim_{T \rightarrow \infty} \left(\frac{1}{2T} \int_{-T}^T |f|^p \right)^{\frac{1}{p}}$$

exist. Roughly, the above norm estimates the average behavior of f for large T . It is easy to see that for any function $f \in L^p(\mathbb{R})$, $\|f\|_{M^p} = 0$. By identifying functions whose difference has zero norm, $(M^p(\mathbb{R}), \|\cdot\|)$ is a Banach space ([3], [9]).

Let B^{pAP} be the class of (Besicovitch) almost periodic functions, i.e. the M^p -closure of the set of trigonometric polynomials $\sum_{k=1}^n a_k e^{it_k(\cdot)}$, $t_k \in \mathbb{R}$. It is known that for $1 < p < \infty$,

B^{pAP} is a nonseparable, reflexive Banach space ($(B^{pAP})^* = B^{p'AP}$, $\frac{1}{p} + \frac{1}{p'} = 1$) and that $B^{pAP} \subseteq W^p(\mathbb{R})$, hence we have

Proposition 2.1. Let $1 < p < \infty$, then $M^p(\mathbb{R})$ contains a nonseparable reflexive subspace.

We can also show that

Proposition 2.2. Let $1 \leq p < \infty$, then $M^p(\mathbb{R})$ contains a subspace isomorphic to ℓ^∞ .

For the extremal structure of the unit sphere $S(M^p(\mathbb{R}))$ of $M^p(\mathbb{R})$, we have

Theorem 2.3. Let $1 < p < \infty$. Then each norm 1 function f in $W^p(\mathbb{R})$ is an extreme point of $S(M^p(\mathbb{R}))$. $S(M^1(\mathbb{R}))$ does not contain any extreme point.

In order to study the duality properties of $M^p(\mathbb{R})$, we introduce the following auxiliary spaces;

$$M^p(\mathbb{R}) = \{f: f \text{ is Borel measurable on } \mathbb{R}, \|f\|_{M^p} = \sup_{T \geq 1} \left(\frac{1}{2T} \int_{-T}^T |f|^p \right)^{\frac{1}{p}} < \infty\}$$

and

$$I^p(\mathbb{R}) = \{f \in M^p(\mathbb{R}) : \lim_{T \rightarrow \infty} \left(\frac{1}{2T} \int_{-T}^T |f|^p \right)^{\frac{1}{p}} = 0\}.$$

It is easy to show that $M^p(\mathbb{R})$ is a Banach space and $I^p(\mathbb{R})$ is a closed subspace of $M^p(\mathbb{R})$.

Theorem 2.4. For $1 < p < \infty$

- (i) $M^p(\mathbb{R})$ is the second dual of $I^p(\mathbb{R})$ and
- (ii) $M^p(\mathbb{R})$ is isometrically isomorphic to $M^p(\mathbb{R})/I^p(\mathbb{R})$.

It follows from Theorem 2.4 that for $1 < p < \infty$,

$$M^p(\mathbb{R})^* = I^p(\mathbb{R})^* \oplus I^p(\mathbb{R})^\perp$$

and $M^p(\mathbb{R})^*$ is isometrically isomorphic to $I^p(\mathbb{R})^\perp$. The concrete representations of functionals on $I^p(\mathbb{R})$ and $M^p(\mathbb{R})$ are given by

Theorem 2.5. Suppose that $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

- (i) If $\ell \in I^p(\mathbb{R})^*$. Then there exists a $\psi \in M^{p'}(\mathbb{R})$ and a countably additive, positive, bounded regular Borel measure μ on $[1, \infty)$ such that for all $f \in I^p(\mathbb{R})$,

$$\langle \ell, f \rangle = \int_1^\infty \left(\frac{1}{2T} \int_{-T}^T f(t)\psi(t)dt \right) d\mu(T). \quad (2.1)$$

- (ii) There exists a (norm) dense subset $D \subseteq M^p(\mathbb{R})^*$ such that ℓ in D can be represented as in (2.1) with $\psi \in M^{p'}(\mathbb{R})$ where μ

is a finite additive, positive, bounded regular Borel measure on $[1, \infty)$, which vanishes on bounded intervals.

§3. The Integrated Lipschitz Class.

For $1 \leq p < \infty$, let $V^p(\mathbb{R})$ denote the class of complex valued Borel measurable functions on \mathbb{R} such that

$$\|g\|_{V^p} = \overline{\lim}_{\varepsilon \rightarrow 0} \left(\frac{1}{2\varepsilon} \int_{-\infty}^{\infty} |g(u+\varepsilon) - g(u-\varepsilon)|^p du \right)^{\frac{1}{p}} < \infty.$$

This class of functions had been studied by Hardy and Littlewood in their investigation of fractional derivatives [5].

Let $g \in V^p(\mathbb{R})$ and let

$$\tilde{g} = \int_0^{\infty} e^{-t} (g - \tau_t g) dt$$

where τ_t is the translation operator defined by

$$(\tau_t f)(x) = f(x+t), \quad f \in V^p(\mathbb{R}).$$

By using the theory of helixes in [13], we can show

Therefore, we have

Proposition 3.2. Let $1 < p < \infty$. If g and g' are in $L^p(\mathbb{R})$, then $\|g\|_{V^p} = 0$.

By identifying functions whose difference has zero norm, we can show that

Theorem 3.3. For $1 < p < \infty$, the normed linear space $V^p(\mathbb{R})$ is complete.

For the case $p = 1$, Nelson [16] showed that $V^1(\mathbb{R})$ is isometrically isomorphic to the space of bounded regular Borel measures on \mathbb{R} , hence the properties of $V^1(\mathbb{R})$ are well known.

54. The Integrated Fourier Transformation.

In proving the identity (1.2), Wiener introduced a fairly general form of Tauberian theorem which applies to functions in $W^2(\mathbb{R})$. In order to consider the Fourier transformation between $M^2(\mathbb{R})$ and $V^2(\mathbb{R})$, we prove another type of Tauberian theorem which applies to the limit supremum of functions at ∞ .

Let M^+ denote the class of positive, Borel measurable functions on $[0, \infty)$ such that $\overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \int_0^T f < \infty$.

Lemma 3.1. Let k be a positive continuous function on $[0, \infty)$. Assume that $\tilde{k}(t) = \sup_{x \geq t} k(x)$ is integrable and $C_1 = \int_0^\infty \tilde{k}(t) dt$. Then for all $f \in M^+$,

$$\overline{\lim}_{T \rightarrow \infty} \int_0^\infty f(Tt)k(t)dt \leq C_1 \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t)dt.$$

Moreover, C_1 is the best possible constant for $f \in M^+$.

Lemma 3.2. Let k be a positive continuous function on $[0, \infty)$, such that $\tilde{k}(t) = \sup_{x \geq t} k(x)$ is integrable. Suppose there is a t_0 satisfying

$$t_0 k(t_0) = \max_{t > 0} t \cdot k(t) (= C_2) \quad \text{and} \quad k(t) \geq k(t_0) \quad \text{for all } t \text{ in } [0, t_0].$$

Then for all $f \in M^+$,

$$C_2 \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t) dt \leq \overline{\lim}_{T \rightarrow \infty} \int_0^\infty f(Tt) k(t) dt.$$

Moreover, C_2 is the best possible constant for $f \in M^+$.

It is easy to show that $M^2(\mathbb{R}) \subseteq L^2(\mathbb{R}, \frac{dt}{1+t^2})$, hence for $f \in M^2(\mathbb{R})$,

the integral

$$\int_{-\infty}^{-1} + \int_1^{\infty} \frac{|f(t)|^2}{t^2} dt$$

exists. This implies that

$$\int_{-\infty}^{-1} + \int_1^{\infty} \frac{f(t) e^{-itu}}{-it} dt$$

converges in $L^2(\mathbb{R}, \frac{dt}{1+t^2})$. Therefore, if $f \in M^2(\mathbb{R})$, we can define

the integrated Fourier transformation $W(f) = g$ as

$$g(u) = \frac{1}{2\pi} \left(\int_{-\infty}^{-1} + \int_1^{\infty} f(t) \frac{e^{-itu}}{-it} dt + \int_{-1}^1 f(t) \frac{e^{-itu} - 1}{-it} dt \right).$$

Now for $h > 0$,

$$\begin{aligned} (\tau_h g - \tau_{-h} g)(u) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \frac{e^{ith} - e^{-ith}}{it} e^{-itu} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \frac{2\sin(ht)}{it} e^{-itu} dt. \end{aligned}$$

Thus $(\tau_h g - \tau_{-h} g)$ is the Fourier transformation of

$$\sqrt{\frac{2}{\pi}} f(t) \frac{\sin(ht)}{t}$$

and the Plancherel theorem implies that

$$\frac{1}{2h} \int_{-\infty}^{\infty} |g(u+h) - g(u-h)|^2 du = \frac{1}{h} \int_{-\infty}^{\infty} |f(t)|^2 \frac{\sin^2 ht}{\pi t^2} dt.$$

Hence,

$$\begin{aligned} \|W(f)\|_{V^2} &= \overline{\lim}_{h \rightarrow 0^+} \frac{1}{h} \int_{-\infty}^{\infty} |f(t)|^2 \frac{\sin^2 ht}{\pi t^2} dt \\ &= \overline{\lim}_{T \rightarrow \infty} \int_{-\infty}^{\infty} |f(Tt)|^2 \frac{\sin^2 t}{\pi t^2} dt. \end{aligned}$$

Letting $k(t) = \frac{2\sin^2 t}{\pi t}$, $t > 0$ and let $\tilde{f}(t) = \frac{1}{2} (|f(t)|^2 + |f(-t)|^2)$, $t > 0$, Lemma 3.1 implies that $W(f) \in V^2(\mathbb{R})$. Moreover, the lemma implies that $W(f) = 0$ for $f \in I^2(\mathbb{R})$. Since $M^2(\mathbb{R}) = M^2(\mathbb{R})/I^2(\mathbb{R})$, W induces a map from $M^2(\mathbb{R})$ into $V^2(\mathbb{R})$. Restating Wiener's theorem in [19], we have

Theorem 4.3. Let $f \in W^2(\mathbb{R})$. Then $\|W(f)\|_{V^2} = \|f\|_{M^2}$.

Our extension of Wiener's theorem is:

Theorem 4.4. The integrated Fourier transformation $W: M^2(\mathbb{R}) \rightarrow V^2(\mathbb{R})$ is an isomorphism with

$$\|W\| = \left(\int_0^{\infty} \tilde{k}(t) dt \right)^{\frac{1}{2}} \quad \text{and} \quad \|W^{-1}\| = \left(\max_{t \geq 0} t \cdot k(t) \right)^{-\frac{1}{2}}$$

where $k(t) = \frac{2\sin^2 t}{\pi t}$, $t > 0$ and $\tilde{k}(t) = \max_{x \geq t} k(x)$.

The two isomorphic constants are direct consequence of Lemma 4.1 and 4.2. Numerically, we find that $\|W\| \approx 1.05$ and $\|W^{-1}\| \approx 1.49$. The surjectivity of W is obtained as follows: for each $g \in V^2(\mathbb{R})$, we may assume that $g \in L^2(\mathbb{R})$ (Proposition 3.1). Let \check{g} denote the inverse Fourier transformation of g and let

$$f(t) = -i \sqrt{2\pi} t \cdot \check{g}(t). \quad (4.1)$$

By direct computation, we can show that $f \in M^2(\mathbb{R})$ and $W(f) = g$.

For $p \neq 2$, we have the following:

Theorem 4.5. For $1 < p < 2$, the integrated Fourier transformation W defines a bounded linear operator form $M^p(\mathbb{R})$ into $V^{p'}(\mathbb{R})$ with

$$\|W\| \leq \left(\int_0^\infty \tilde{k}(t) dt \right)^{\frac{1}{p}}$$

where $k(t) = \left| \frac{2\sin^p t}{\pi t^p} \right|$, $t > 0$.

§5. The Convolution Operators.

We call a function f in $M^p(\mathbb{R})$ regular if

$$\lim_{T \rightarrow +\infty} \int_T^{T+a} |f|^p = 0 \quad \text{for any } a > 0.$$

Let $M_r^p(\mathbb{R})$ denote the set of regular functions in $M^p(\mathbb{R})$ and let $M_r^p(\mathbb{R}) = M_r^p(\mathbb{R})/I^p(\mathbb{R})$. Then $M_r^p(\mathbb{R})$ is a closed subspace of $M^p(\mathbb{R})$ and $W^p(\mathbb{R}) \subseteq M_r^p(\mathbb{R})$.

Let M denote the set of bounded regular Borel measure on R and let M_1 be the subspace of measures with bounded support. For $\mu \in M_1$, we define the convolution operator $\phi_\mu : M^p(R) \rightarrow M^p(R)$ by

$$\phi_\mu(f) = \mu * f, \quad f \in M^p(R).$$

Note that ϕ_μ also defines an operator from $L^p(R)$ into $L^p(R)$.

Restricting ϕ_μ to $M_r^p(R)$ yields

Proposition 5.1. For $\mu \in M_1$, the operator $\phi_\mu : M_r^p(R) \rightarrow M_r^p(R)$ satisfies

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_R |(\chi_{[-T,T]} \phi_\mu - \phi_\mu \chi_{[-T,T]})f|^p = 0.$$

By using this, we can show that for $f \in M_r^p(R)$

$$\begin{aligned} \|\phi_\mu(f)\|_{M^p} &= \overline{\lim}_{T \rightarrow \infty} \left(\frac{1}{2T} \int_{-T}^T |\phi_\mu(f)|^p \right)^{\frac{1}{p}} \\ &= \overline{\lim}_{T \rightarrow \infty} \left(\frac{1}{2T} \int_R |\phi_\mu(\chi_{[-T,T]} f)|^p \right)^{\frac{1}{p}} \\ &\leq \|\phi_\mu\|_{L^p} \cdot \|f\|_{M^p}. \end{aligned} \tag{5.1}$$

Conversely, if for any $\epsilon > 0$, there exists an $f \in L^p(R)$ such that

$$\|\phi_\mu(f)\|_{L^p} \geq \|\phi_\mu\|_{L^p} \cdot \|f\|_{L^p} - \epsilon$$

we can construct an \tilde{f} in $M_r^p(R)$ such that

$$\|\phi_\mu(\tilde{f})\|_{M^p} \geq \|\phi_\mu\|_{L^p} \cdot \|\tilde{f}\|_{M^p} - \epsilon.$$

This fact and (5.1) imply that

$$\|\phi_\mu\|_{M^p} = \|\phi_\mu\|_{L^p} \quad \text{for } \mu \in M_1.$$

It is known that for $\mu \in M$ and for $p = 1$, $\|\phi_\mu\|_{L^1} = \|\mu\|$; for $p = 2$, $\|\phi_\mu\|_{L^2} = \|\hat{\mu}\|_\infty$ where $\hat{\mu}$ is the Fourier-Stieltjes transformation of μ ; and for $1 < p < \infty$, $p \neq 2$,

$$\|\hat{\mu}\|_\infty \leq \|\phi_\mu\|_{L^p} \leq \|\mu\|.$$

If $\mu \in M$, then there exists a sequence of $\{\mu_n\}$ in M_1 which converges to μ , hence $\{\phi_{\mu_n}\}$ is a Cauchy sequence of operators in $M^p(\mathbb{R})$ and converges to the operator ϕ_μ .

Theorem 5.2. For $1 < p < \infty$ and for $\mu \in M$, the convolution operator

$$\phi_\mu : M_r^p(\mathbb{R}) \rightarrow M_r^p(\mathbb{R}) \quad \text{is well defined and } \|\phi_\mu\|_{M^p} = \|\phi_\mu\|_{L^p}.$$

Let $g \in V^2(\mathbb{R})$ and let g be the inverse Fourier transformation of g , then the function defined by $f(t) = i\sqrt{2\pi} t \cdot g(t)$ is in $M^2(\mathbb{R})$ and $W(f) \doteq g$ (see (4.1)). For each $\mu \in M_1$, we have

$$\phi_\mu(f)(t) = -i\sqrt{2\pi} t \cdot \phi_\mu(g)(t) = -i\sqrt{2\pi} t \cdot (\mu * g(t)) = -i\sqrt{2\pi} t \cdot (\hat{\mu} \cdot g)(t).$$

This implies that for $\mu \in M_1$,

$$W(\mu * f) = W(\phi_\mu f) = \hat{\mu} \cdot g = \hat{\mu} \cdot Wf.$$

By using a limit argument, we can show that:

Theorem 5.4. For each $\mu \in M$, the convolution operator

$$\phi_\mu : M_r^2(\mathbb{R}) \rightarrow M_r^2(\mathbb{R}) \quad \text{defines the multiplier operator } \psi_\mu : V_r^2(\mathbb{R}) \rightarrow V_r^2(\mathbb{R})$$

$(V_{\mathbb{R}}^2 = W(M_{\mathbb{R}}^2))$ which satisfies

$$W(\Phi_{\mu}(f)) = \Psi_{\hat{\mu}}(W(f))$$

Moreover, $\|\Phi_{\mu}\| = \|\hat{\mu}\|_{\infty}$ and

$$C^{-1} \|\hat{\mu}\|_{\infty} \leq \|\Psi_{\hat{\mu}}\|_{\mathcal{V}_2} \leq \|\hat{\mu}\|_{\infty}, \quad \mu \in M$$

where $C = \|\bar{W}\| \cdot \|W^{-1}\|$.

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University of Pittsburgh
Pittsburgh, PA 15260